

Comments on “Charged particle dynamics in the field of slowly rotating compact star”

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Abstract

B.M. Mirza [1] presented a solution of coupled Einstein-Maxwell equations for a slowly rotating neutron star; however his derivations had some errors and implicit assumptions that rendered the solution invalid. We point out the errors and present a mathematically consistent solution. The resulting solution is also physically consistent as it remains finite in the no rotation limit, whereas Mirza’s solution diverges for zero rotation.

Slowly rotating neutron stars were first investigated in 1967 as a slow rotation approximation ($0 < a < 1, O(a^2)$) to the Kerr metric [2] and have astrophysical relevance as most of the observed pulsars are actually slowly rotating relative to the speed of light (one fifth for a millisecond pulsar) [3]. Charged particle dynamics around such stars is investigated by constructing the corresponding Einstein-Maxwell equations assuming the slow rotation approximation [4]. Later Mirza [1] solved the same model using an ansatz. The chosen ansatz was dimensionally inconsistent and yielded an unphysical answer for the fields which gives a divergent expression for the radiation emitted in the no rotation limit. We prove that his chosen ansatz, even after dimensional modification, gives a divergent result in the no rotation limit. We then suggest an ansatz that avoids divergences in any limit and is physically more meaningful.

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The spacetime exterior to a slowly rotating neutron star is given by the slow rotation approximation to the Kerr metric:

$$ds^2 = -e^{2\Phi(r)} dt^2 - 2\omega(r)r^2 \sin^2 \theta dt d\varphi + r^2 \sin^2 \theta d\varphi^2 + e^{-2\Phi(r)} dr^2 + r^2 d\theta^2, \quad (1)$$

where

$$e^{2\Phi(r)} = \left(1 - \frac{2M}{r}\right), \quad (2)$$

and

$$\omega(r) \equiv \frac{d\varphi}{dt} = -\frac{g_{t\varphi}}{g_{\varphi\varphi}}, \quad (3)$$

is the angular speed of a freely falling frame brought into rotation by frame dragging. Here and in what follows the Greek indices run as t, r, θ , and φ respectively and we use gravitational units in which $G = 1 = c$. The general relativistic form of the Maxwell equations is:

$$F_{\alpha\beta,\gamma} + F_{\beta\gamma,\alpha} + F_{\gamma\alpha,\beta} = 0, \quad (4)$$

$$\left(\sqrt{-g}F^{\alpha\beta}\right)_{,\beta} = 4\pi\sqrt{-g}J^\alpha, \quad (5)$$

where g is the determinant of the metric tensor $g_{\alpha\beta}$ given by the Einstein equations and J^α is 4-vector current density. Here $F_{\alpha\beta}$ is the generalized electromagnetic field tensor for an ideal fluid given by a unique tensorial expression:

$$F_{\alpha\beta} = u_\alpha E_\beta - u_\beta E_\alpha + \eta_{\alpha\beta\gamma\delta} u^\gamma B^\delta, \quad (6)$$

where $\eta_{\alpha\beta\gamma\delta}$ is the totally skew tensor, E_α and B^α are the electric and magnetic fields and u^α is the unit velocity 4-vector [5]. In general J^α is the sum of two terms corresponding to a convection and a conduction current:

$$J^\alpha = \epsilon u^\alpha + \sigma u_\beta F^{\beta\alpha} \quad (7)$$

where ϵ is the proper charge density, σ is the conductivity of the fluid. For a zero angular momentum observer (ZAMO) u_r and u_θ vanish, and using $u^\alpha u_\alpha = -1$, the components of the 4-velocity vector are:

$$u^\alpha = e^{-\Phi(r)}(1, 0, 0, \omega(r)), \quad u_\alpha = e^{\Phi(r)}(-1, 0, 0, 0). \quad (8)$$

The electromagnetic field outside the neutron star is now determined by eqs.(4) and (5). To solve this system of equations let us assume Mirza's ansatz modified to maintain dimensional consistency, for the electric field \mathbf{E} :

$$E_r(r, \theta) \equiv k_1 E_\theta(r, \theta) \equiv k_2 E_\varphi(r, \theta) = R_E(r) \Theta_E(\theta), \quad (9)$$

where k_1 and k_2 are some dimensional constants. For the magnetic field \mathbf{B} , the corresponding ansatz will be

$$B_r(r, \theta) \equiv k_3 B_\theta(r, \theta) \equiv k_4 B_\varphi(r, \theta) = R_B(r) \Theta_B(\theta), \quad (10)$$

where k_3 and k_4 are some dimensional constants. Further, following Mirza, we take a constant angular speed of rotation, ω_\circ .

Solving eqs.(4) and (5) using eqs.(9) and (10), we get

$$E_r \equiv k_1 E_\theta \equiv k_2 E_\varphi = R_E(r) \Theta_E(\theta) = \frac{A_1}{\omega_\circ r^2 \sin \theta}, \quad (11)$$

$$B_r \equiv k_3 B_\theta \equiv k_4 B_\varphi = R_B(r) \Theta_B(\theta) = \frac{A_2}{\omega_\circ r^2 \sin \theta}, \quad (12)$$

where A_1 and A_2 are arbitrary constants. It can easily be seen that in the no rotation limit i.e $\omega_\circ \rightarrow 0$, the electric and magnetic fields become infinite, which cannot be true, because it yields infinite energy radiated by a non-rotating neutron star. Further, the Poynting vector $\mathbf{S} = \mathbf{E} \times \mathbf{B}$ gives the momentum flux and hence the energy radiated ε . It yields $\varepsilon \propto \frac{1}{\omega_\circ^2}$ where the proportionality factor depends upon $k_1 \dots k_4$ and (r, θ) . If it is non zero, the radiated energy diverges as $\omega_\circ \rightarrow 0$ which is impossible, and if $\varepsilon = 0$, the rotating star would not radiate and hence would not be a model for a pulsar. Infact, in general, for a rotating neutron star $\frac{d\varepsilon}{dt} \propto \omega^6$ [3]. Hence Mirza's ansatz does not work even after correcting for dimensions.

We present another ansatz that avoids the above mentioned impossibilities.

$$\mathbf{E} = (E_r, 0, E_\varphi), \mathbf{B} = (0, B_\theta, 0), \quad (13)$$

where \mathbf{E} and \mathbf{B} has r and θ dependence only. To solve eqs.(4) and (5) we shall use the following separation ansatz for electric and magnetic fields

$$E_r(r, \theta) \equiv k_5 E_\varphi(r, \theta) = R_E(r) \Theta_E(\theta), \quad (14)$$

$$B_\theta(r, \theta) = R_B(r) \Theta_B(\theta). \quad (15)$$

Solving eqs.(4) and (5) using eqs.(13), (14) and (15) we get

$$E_r \equiv k_5 E_\varphi = R_E(r) \Theta_E(\theta) = \frac{A_3}{r^2 u^t}, \quad (16)$$

$$B_\theta = R_B(r) \Theta_B(\theta) = \frac{A_4 \omega_\circ}{u^t \sin \theta}, \quad (17)$$

where A_3 and A_4 are arbitrary constants. Hence, \mathbf{E} and \mathbf{B} remain finite in the no-rotation limit and the radiation vanishes for a non-rotating object, as required.

There will be several other ansatz that would yield solutions to the system of equations but care must be taken to avoid non-physical solutions such as Mirza obtained. As an extension to the problem, one can solve the above system using the source terms i.e. $j^\alpha \neq 0$ and deduce some physically interesting realizable results.

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